MATH 2040A Tutorial Week 9

§6.1 Inner products

19. Let V be an inner product space. Prove that

a) $\|x \pm y\|^2 = \|x\|^2 \pm 2\,Re(x,y) + \|y\|^2$ for all $x \in V$, where $Re(x,y) :=$ real part of (x,y)

b) $|\,\|x\| - \|y\|\,| \le \|x-y\|$ for all $x, y \in V$,

Solution: a) $\|x+y\|^2 = \|x\|^2 + \langle x,y\rangle + \langle y,x\rangle + \|y\|^2$

$= \|x\|^2 + \langle x,y\rangle + \overline{\langle x,y\rangle} + \|y\|^2 = \|x\|^2 + 2Re\langle x,y\rangle + \|y\|^2$

$\|x-y\|^2 = \|x\|^2 - \langle x,y\rangle - \langle y,x\rangle + \|y\|^2$

$= \|x\|^2 - \langle x,y\rangle - \overline{\langle x,y\rangle} + \|y\|^2 = \|x\|^2 - 2Re\langle x,y\rangle + \|y\|^2$

b) $\|x-y\| + \|y\| \ge \|x\|$ $\quad \|x-y\| + \|x\| \ge \|y\|$ (Triangle inequality)

$\Rightarrow |\,\|x\| - \|y\|\,| \le \|x-y\|$ for all $x,y \in V$

20. Let V be an inner product space over F. Prove ~~that~~ the polar identities: For all $x,y \in V$

a) $\langle x,y\rangle = \frac{1}{4}\|x+y\|^2 - \frac{1}{4}\|x-y\|^2$ if $F = \mathbb{R}$.

b) $\langle x,y\rangle = \frac{1}{4}\sum_{k=1}^{4} i^k \|x + i^k y\|^2$ if $F = \mathbb{C}$, where $i^2 = -1$.

Solution: a) By 19, a) : $\|x \pm y\|^2 = \|x\|^2 \pm 2\,Re(x,y) + \|y\|^2$

$Re\langle x,y\rangle = \langle x,y\rangle$ $\quad (F = \mathbb{R})$

$\Rightarrow RHS = \frac{1}{4}(4\,Re\langle x,y\rangle) = Re\langle x,y\rangle = \langle x,y\rangle$

b) $\|x + i^k y\|^2 = \|x\|^2 + 2\,Re(\overline{i^k}\langle x,y\rangle) + \|y\|^2$

$a := Re\langle x,y\rangle \quad b = Im\langle x,y\rangle$

$RHS = \frac{1}{4}\left(\left(\sum_{k=1}^{4} i^k\right)\|x\|^2 + 2\sum_{k=1}^{4} i^k Re(\overline{i^k}\langle x,y\rangle) + \left(\sum_{i=1}^{4} i^k\right)\|y\|^2 \right)$

$= \frac{1}{2}(bi + (-a)(-1) + (-b)(-i) + a) = a + bi = \langle x,y\rangle$

§6.2 Gram Schmidt Process

2. Apply the GS Process to S to get an orthogonal basis for span S. Normalize it to get an orthonormal one, β. Compute the Fourier coefficients of x relative to β.

a) $V = \mathbb{R}^3$, $S = \{(1,0,1), (0,1,1), (1,3,3)\}$ $\quad x = (1,1,2)$

Solution: • $w_1 := (1,0,1)$, $w_2 := (0,1,1)$, $w_3 := (1,3,3)$.

• $v_1 = w_1$. $\quad v_2 = w_2 - \frac{\langle w_2, v_1\rangle}{\|v_1\|^2} v_1 = w_2 - \frac{1}{2}v_1 = (-\frac{1}{2}, 1, \frac{1}{2})$

$v_3 = w_3 - \frac{\langle w_3, v_1\rangle}{\|v_1\|^2} v_1 - \frac{\langle w_3, v_2\rangle}{\|v_2\|^2} v_2 = w_3 - \frac{4}{2}v_1 - \frac{8}{3}v_2 = (\frac{1}{3}, \frac{1}{3}, -\frac{1}{3})$

• Normalize $v_1\, v_2\, v_3$: $u_1 = (\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}})$ $\quad u_2 = (-\frac{1}{\sqrt{6}}, \frac{4}{\sqrt{6}}, \frac{1}{\sqrt{6}})$ $\quad u_3 = (\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}})$

• $a_1 u_1 + a_2 u_2 + a_3 u_3 = x$ $\quad$ solve this system of equations : $x = \frac{3}{\sqrt{2}}u_1 + \frac{3}{\sqrt{6}}u_2 + 0 \cdot u_3$

Or, by Thm 6.5, $a_i = \langle x, u_i \rangle$ which agree with those obtained by the first method.

3. In $\mathbb{R}^2$, let $\beta = \left\{\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right), \left(\frac{1}{\sqrt{2}}, \frac{-1}{\sqrt{2}}\right)\right\}$ Find the Fourier coefficients of $(3, 4)$ relative to β.

Solution: $\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right) \cdot \left(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right) = 0$ $\quad \left\|\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)\right\| = \left\|\left(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right)\right\| = 1$

$\Rightarrow \beta$ is orthonormal

$(3,4) \cdot \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right) = \frac{7}{\sqrt{2}}$ $\quad (3,4)\left(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right) = -\frac{1}{\sqrt{2}}$

4. $S = \{(1, 0, i), (1, 2, 1)\}$ in $\mathbb{C}^3$. Compute $S^\perp$

Solution: $(a, b, c) \cdot (1, 0, i) = a - ci = 0$ $\quad$ Solve this system

$(a, b, c) \cdot (1, 2, 1) = a + 2b + c = 0$

$\Rightarrow S^\perp = \mathrm{Span}\left\{\left(i, -\frac{1}{2}(1+i), 1\right)\right\}$

6. Let V be an inner product space. W a fin. dim. subspace of V. If $x \notin W$. prove that $\exists y \in V$ s.t. $y \in W^\perp$, but $\langle x, y \rangle \neq 0$.

Solution: $X = \mathrm{Span}\{W, x\}$. It is of fin. dim.

$\xrightarrow{\text{Thm 6.6}}$ x can be uniquely written as $u + v$ with $u \in W$, $v \in V^\perp$.

$x \notin W \Rightarrow v \neq 0$. $\quad y := v$. then $\langle x, y \rangle = \langle v, v \rangle + \langle u, v \rangle = \|v\|^2 > 0$

7. Let β be a basis for a subspace W of an inner product space V, and let $z \in V$. Prove that $z \in W^\perp$ iff $\langle z, v \rangle = 0$ for every $v \in \beta$.

Solution: $\Rightarrow$ By definition of orthogonal complement.

$\Leftarrow$ We want to check that $\langle z, v \rangle = 0$ for all $v \in W$.

v can be written as $\sum_{i=1}^{k} a_i v_i$ $\quad a_i$ scalar, $v_i \in \beta$

$\langle z, \sum_{i=1}^{k} a_i v_i \rangle = \sum_{i=1}^{k} \bar{a}_i \langle z, v_i \rangle = 0$

9. $W = \text{span}(\{(i, 0, 1)\})$ in $\mathbb{C}^3$. Find orthonormal basis for W and $W^\perp$.

Solution: For W, $\{\frac{1}{\sqrt{2}}(i, 0, 1)\}$

For $W^\perp$, we ~~first~~ find the basis for the null space of the following system of equations

$(a, b, c) \cdot (i, 0, 1) = -ai + c = 0$.

$\{(1, 0, i), (0, 1, 0)\}$ is an orthogonal basis for this space.

So $\{\frac{1}{\sqrt{2}}(1, 0, i), (0, 1, 0)\}$ is the orthonormal basis for $W^\perp$.

10. Let W be ~~an~~ a fin. dim. subspace of an inner product space V. Prove that there exists a projection T on W along $W^\perp$ that satisfies $N(T) = W^\perp$. In addition Prove that $\|T(x)\| \le \|x\|$ for all $x \in V$.

Solution: Thm 6.6 $\Rightarrow V = W \oplus W^\perp \Rightarrow \exists$ projection T on W along $W^\perp$.

$\forall x \in V$ can be written uniquely as $u+v$ $u \in W$ $v \in W^\perp$

$T(x) := u$. Then $Null(T) = W^\perp$.

$\|x\|^2 = \|u\|^2 + \|v\|^2 \ge \|u\|^2 = \|T(x)\|^2$ $(\langle u, v \rangle = 0)$

$\Rightarrow \|T(x)\| \le \|x\|$

12. Prove that for any matrix $A \in M_{m\times n}(F)$, $(R(L_{A^*}))^\perp = N(L_A)$

Solution: If $x \in (R(L_{A^*}))^\perp$, then x is orthogonal to A^*y for all $y \in \mathbb{F}^m$.

$0 = \langle x, A^*y \rangle = \langle Ax, y \rangle$. for all y

$\Rightarrow Ax = 0 \Rightarrow x \in N(L_A)$

Conversely. $x \in N(L_A) \Rightarrow Ax = 0 \Rightarrow \langle x, A^*y \rangle = \langle Ax, y \rangle = 0$.

5. V fin dim inner product space / F.

a) Parseval's Identity. $\{v_1 \cdots v_n\}$ orthonormal basis for V. For any $x, y \in V$ prove that
$\langle x, y \rangle = \sum_{i=1}^{n} \langle x, v_i \rangle \langle y, v_i \rangle$

b) Prove that if β is an orthonormal basis for V with inner product $\langle , \rangle$, then for any $x, y \in V$, $\langle \phi_\beta(x), \phi_\beta(y) \rangle' = \langle [x]_\beta, [y]_\beta \rangle' = \langle x, y \rangle$

$\langle \ \rangle'$ standard inner product on F^n.

Solution: a) Thm 6.5 $\Rightarrow$ $x = \sum_{i=1}^{n} \langle x, v_i \rangle v_i$, $y = \sum_{j=1}^{n} \langle y, v_j \rangle v_j$.

Thus, $\langle x, y \rangle = \sum_{i,j} \langle \langle x, v_i \rangle v_i, \langle y, v_j \rangle v_j \rangle$

$$= \sum_{i=1}^{n} \langle \langle x, v_i \rangle v_i, \langle y, v_i \rangle v_i \rangle = \langle x, v_i \rangle \overline{\langle y, v_i \rangle}$$

b) By definition. $\langle [x]_\beta, [y]_\beta \rangle' = \sum_{i=1}^{n} \langle x, v_i \rangle \overline{\langle y, v_i \rangle}$

16. a) Bessel's Inequality. V - inner product space. $S = \{v_1 \cdots v_n\}$ an orthonormal subset of V. Prove that for any $x \in V$ we have $\|x\|^2 \geq \sum_{i=1}^{n} |\langle x, v_i \rangle|^2$.

Solution: $W := \text{Span } S$. - fin dim.

$u \in W$, then 15 a) $\Rightarrow$ $\|u\|^2 = \sum_{i=1}^{n} |\langle u, v_i \rangle|^2$.

For a fixed x, $W' = \text{Span}(W \cup \{x\})$ is fin dim.

10 $\Rightarrow$ $T(x) \in W$ and $\|T(x)\| \leq \|x\|$.

$\Rightarrow$ $\|x\|^2 \geq \|T(x)\|^2 = \sum_{i=1}^{n} |\langle T(x), v_i \rangle|^2$.

$x = T(x) + y$ by def. of T. $y \in W^\perp$.

$\Rightarrow$ $\langle x, v_i \rangle = \langle T(x), v_i \rangle + \langle y, v_i \rangle = \langle T(x), v_i \rangle$

18. $V = \mathbf{C}([-1,1])$ (Continuous functions on $[-1,1]$) with inner product $\langle f, g \rangle = \int_{-1}^{1} f(t)g(t)dt$

We (resp. Wo) subspaces of V of even (odd) functions. Prove that $W_e^\perp = W_o$

Solution: f - odd function. For every even function g, fg is odd.

$\Rightarrow \langle f, g \rangle = 0$, $\Rightarrow$ $W_e^\perp \supset W_o$

For any function h, $h = f + g$. $f \in W_e$, $g \in W_o$.

If $h \in W_e^\perp$, $\Rightarrow$ $0 = \langle h, f \rangle = \langle f, f \rangle + \langle g, f \rangle = \|f\|^2$

$\Rightarrow f = 0$, $h = g \in W_o$.